Group Velocity & Phase Velocity

Quantum Mechanics

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Consider a plane monochromatic wave propagating in '+x' direction.

$$\Psi(x,t) = A\, e^{i(kx - \omega t)} \qquad k = \frac{2\pi}{\lambda}$$

What is the forward speed of this wave?

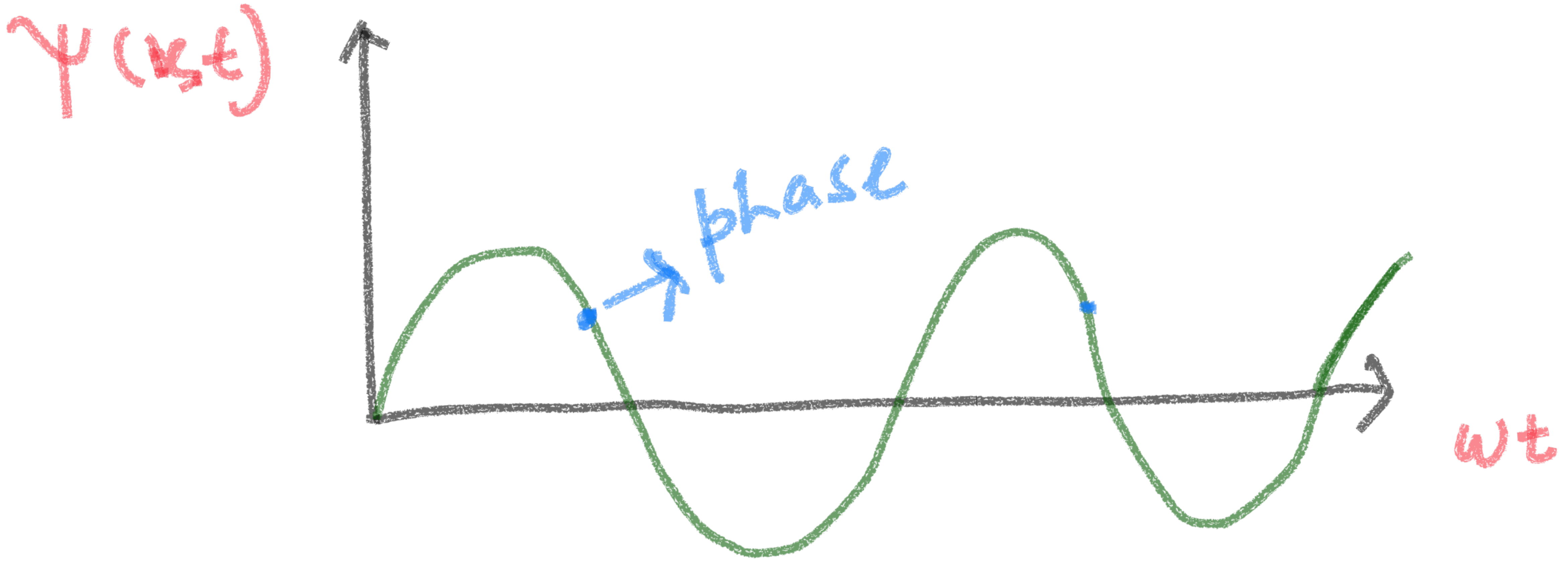
The phase of this wave is $\theta = kx - \omega t$

choose a value of θ, then the velocity of that constant (definite) phase moves with $\frac{d\theta}{dt} = 0 \Rightarrow \omega = \frac{dx}{dt} k = v_p k$

As time evolves the phase moves with a speed $v_p = \frac{\omega}{k} \longrightarrow$ Phase velocity in x-direction

Thus the Phase Velocity of a single wave is the velocity with which a definite phase of the wave propagates.

$$\boxed{v_p = \frac{\omega}{K}}$$



The monochromatic waves are same in shape w.r.t. time and space

Interesting changes happen when we superpose two or more such monochromatic waves.

Group Velocity

Consider two plane waves with same amplitudes 'A' but different frequencies ω_1 and ω_2.

$$\psi_1 = A\cos(\omega_1 t - k_1 x) \; ; \; \psi_2 = A\cos(\omega_2 t - k_2 x)$$

If we superpose them, we get

$$\psi = \psi_1 + \psi_2 = 2A\cos\left(\frac{\omega_2 - \omega_1}{2} t - \frac{k_2 - k_1}{2} x\right) \times \cos\left(\frac{\omega_1 + \omega_2}{2} t + \frac{k_1 + k_2}{2} x\right)$$

$$= \tilde{A}\cos\left(\frac{\omega_1 + \omega_2}{2} t + \frac{k_1 + k_2}{2} x\right)$$

The amplitude: $\tilde{A} = 2A \cos\left(\frac{\omega_2 - \omega_1}{2} t - \frac{k_2 - k_1}{2} x\right)$

If the frequencies are very close to each other

then $\omega_2 \cong \omega_1 + \Delta\omega$

and $k_2 \cong k_1 + \Delta k$

Then the phase of Ψ is $\bar{\Theta} = \frac{2\omega_1 + \Delta\omega}{2} t - \frac{2k_1 + \Delta k}{2} x$

the phase has a velocity $\Rightarrow \frac{2\omega_1 + \Delta\omega}{2k_1 + \Delta k}$

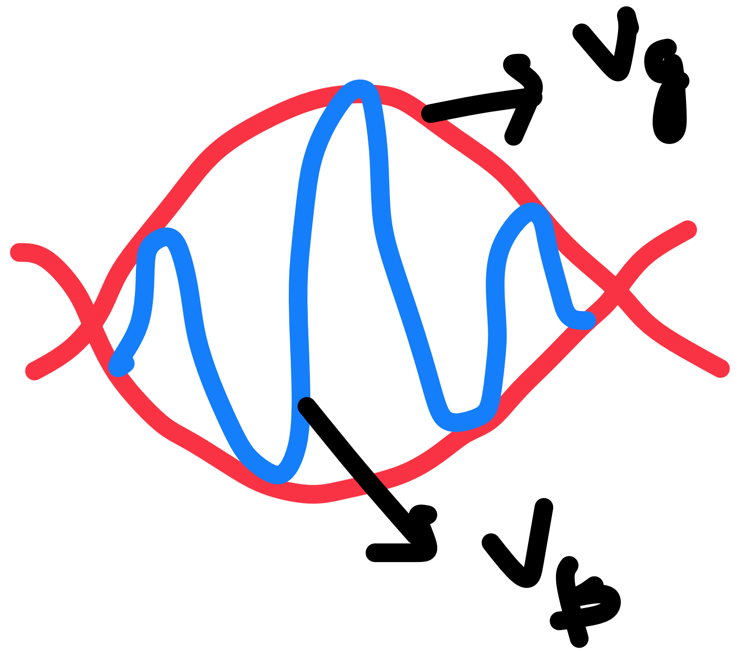
as $\Delta\omega \to 0 \quad \cong \frac{\omega}{k}$

Now the amplitude also has a modulation. It emphasises the 'envelop' of the wave group.

As $\omega_2 - \omega_1 \sim \Delta\omega$

$k_2 - k_1 \sim \Delta k$

The velocity of the envelop is



Group velocity $\rightarrow$ $\boxed{v_g = \frac{\Delta\omega}{\Delta k} \rightarrow \frac{d\omega}{dk}}$

$$v_g = \frac{d\omega}{dk} \Rightarrow \frac{d}{dk}(v_p k) = v_p + k\frac{dv_p}{dk}$$

$$\because \frac{dk}{d\lambda} = -\frac{2\pi}{\lambda^2} \Rightarrow \boxed{v_g = v_p - \lambda\frac{dv_p}{d\lambda}}$$

For a non-dispersive medium v_p is independent of λ.

$$\omega = v_p K \quad \text{so} \quad \frac{d\omega}{dK} = v_p = v_g$$

$v_p v_g = c^2$ Relation (Group and Phase velocity of de Broglie wave)

$$\omega = 2\pi \nu$$

$$E = h\nu$$

Relativistic relation : $E = mc^2$

$$\Rightarrow \quad \omega = \frac{2\pi mc^2}{h} \quad \text{and} \quad K = \frac{2\pi}{\lambda}$$

$$v_p = \frac{\omega}{k} = \frac{2\pi\, mc^2}{h}\,\frac{\lambda}{2\pi} = \frac{m_0 c^2}{\sqrt{1-v^2/c^2}}\,\frac{\lambda}{h}$$

de-broglie relation $\quad p = \frac{h}{\lambda}$

$$\Rightarrow \quad v_p = \frac{mc^2}{p} = \frac{c^2}{v}$$

$$v_g = \frac{d\omega}{dv}\,\frac{dv}{dk} = \frac{\dfrac{2\pi m_0}{3\sqrt{1-v^2/c^2}}\,\dfrac{v}{c^2}\cdot c^2}{\dfrac{2\pi m_0}{3\sqrt{1-v^2/c^2}}}$$

$$= v$$

$$\because \omega = \frac{2\pi\, m_0 c^2}{\sqrt{1-v^2/c^2}}\,\frac{1}{h}$$

$$K = \frac{2\pi p}{h} = \frac{2\pi m_0 v}{h\sqrt{1-v^2/c^2}}$$

$$\therefore \boxed{v_p = \frac{c^2}{v_g}}$$

Non Relativistic Relation :-

$$E = \frac{p^2}{2m} \quad ; \quad p = \hbar k \qquad \text{(Free particle)}$$

$$E = \frac{\hbar^2 k^2}{2m} = \hbar\omega$$

$$\therefore \quad \omega = \frac{\hbar k^2}{2m}$$

$$\Rightarrow \frac{dE}{dp} = \frac{d\omega}{dk} = \frac{\hbar k}{m} = \frac{p}{m} = v_g \rightarrow \text{particle velocity}$$

For relativistic case :- $E = mc^2 = pc$

$$\Rightarrow E^2 = p^2c^2 \Rightarrow \frac{dE}{dp} = \frac{pc^2}{E} = \frac{p}{m} = \frac{mv}{m} = v$$

again the particle velocity.

Wave packet and particle interpretation

$$f(x,t) = \int_{-\infty}^{\infty} a(k)\, e^{-i(kx - \omega t)}\, dk$$

Let, $a(k)$ is contributing within $k_0 - \Delta k,\ k_0 + \Delta k$

The phase $\phi_k(k) = kx - \omega t$

Let, the stationary pt of ϕ_k is k_0

$$\therefore \left.\frac{d\phi_k}{dk}\right|_{k=k_0} = x - \omega'(k_0)\,t = 0 \Rightarrow x = v_g t$$

$x = v_g t \Rightarrow$ The maxima of the phase is traveling with the speed v_g.

$$\omega(k) \simeq \omega(k_0) + (k-k_0)\, v_g + \left.\frac{d^2\omega}{dk^2}\right|_{k_0} \frac{(k-k_0)^2}{2}$$

$$f(x,t) = e^{-i(k_0 x - \omega(k_0) t)} \underbrace{\int_{-\infty}^{\infty} a(k)\, e^{-i(k-k_0)(x-v_g t)}\, dk}$$

using $\frac{p}{k} = \hbar$

Assuming $\frac{(k-k_0)^2}{2}\, t \left.\frac{d^2\omega}{dk^2}\right|_{k=k_0} \ll 1$

$\Rightarrow$

$$\boxed{f(x,t) = \underbrace{e^{-\frac{i}{\hbar}[p_0 x - E(p_0) t]}}_{\text{plane wave}} \times \underbrace{g(x,t)}_{\text{modulating envelope}}}$$

$$g(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \tilde{a}(p)\, e^{-\frac{i}{\hbar}(p-p_0)(x-v_g t)}\, dp$$

This envelop propagates without change in shape with group velocity v_g.

Fourier transform :-
$$\tilde{a}(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} g(x,t)\, e^{\frac{i}{\hbar}px}\, dx$$

$\tilde{a}(p,t)$ → momentum space wavefn

$g(x,t)$ → position space wavefunction

Why $\psi(x,t)$ must be complex?

If $\psi(x,t)$ describes a quantum particle w.r.t. an origin O. Then if we shift the origin by 'a' then $\psi(x+a,t)$ should describe the same system.

If $\psi(x,t)$ is real wave such as

$\psi(x,t) = A \sin(kx - \omega t)$ $\quad \psi^2 = A^2 \sin^2(kx - \omega t)$

the probability $\quad \because$ probability can't be -ve

But $\psi^2(x,t) \neq \psi^2(x+a,t)$ $\longrightarrow$ check!

But if $\psi(x,t) = A e^{i(kx-\omega t)}$; A real

then $\psi(x+a,t) = A e^{ika} e^{i(kx-\omega t)}$

and, $\psi^*\psi\big|_{(x,t)} = A^2 = \psi^*\psi\big|_{x+a,t}$

Then $\psi(x,t)$ better be complex.